Probing an unknown elastic body with waves that scatter once: An inverse problem in anisotropic elasticity.

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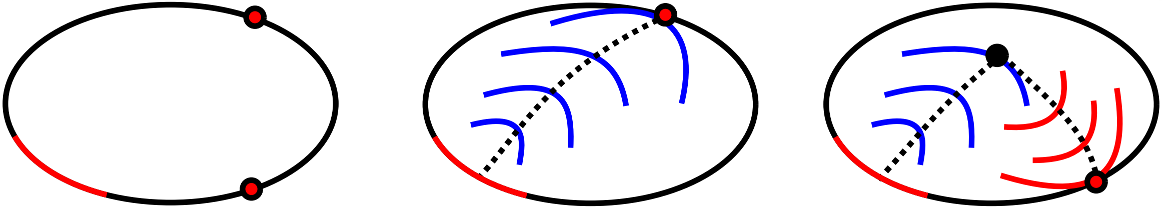
Probing with waves

Problem

Can we recover some information about an **unknown** *object via observing waves that*

- *travel through the object?*
- *scatter once inside before leaving the object?*



- We send and receive waves anywhere on the surface of an unknown object
- Waves do not reflect from the surface, but exit the object

Our goal is to recover physical quantities that are coordinate independent!

1. Sound waves: Riemannian geometry
2. Elastic waves: Finsler geometry
3. Broken ray tomography: Inverse problem of broken scattering relation

Contents

Principal symbol of a partial differential operator

Fourier transfrom: $\mathcal{F}(D^\alpha u)(p) = p^\alpha(\mathcal{F}u)(p)$,

$$p \in \mathbb{R}^3,\ \alpha \in \mathbb{N}^3, \quad |\alpha| = \sum_{i=1}^{3} \alpha_i, \quad \mathrm{D}^\alpha = (-i)^{|\alpha|} \prod_{i=1}^{3} \partial_{x_i}^{\alpha_i}, \quad p^\alpha = \prod_{i=1}^{3} p_i^{\alpha_i}.$$

Thus we can think Partial Differential Operators of order $k \in \mathbb{N}$

$$P(x, \mathrm{D}) = \sum_{|\alpha| \leq k} c_\alpha(x) \mathrm{D}^\alpha, \quad \text{as polynomials} \quad P(x,p) := \sum_{|\alpha| \leq k} c_\alpha(x) p^\alpha,\ (x,p) \in T^*\mathbb{R}^3$$

Principal symbol:

$$G(x,p) = \sum_{|\alpha| = k} c_\alpha(x) p^\alpha, \quad \text{if } c_\alpha \in C^\infty(\mathbb{R}^3) \ \Rightarrow\ G \in C^\infty(T^*\mathbb{R}^3)$$

2nd order elliptic operators and geometry:

Consider $G(x,p)$ as a smooth Hamiltonian function on $T^*\mathbb{R}^3$.

Integral curve: $$(x(t),p(t)) \in T^*\mathbb{R}^3, \quad \dot{x}(t) = \frac{\partial}{\partial p}G(x,p), \quad \dot{p}(t) = -\frac{\partial}{\partial x}G(x,p).$$

$(x(t),p(t))$ is a bi-characteristic curve if $G(x(t),p(t)) = 0$ for any $t \in \mathbb{R}$.

Operator $P(x)$ of order k is elliptic if the principal symbol $G(x,p)$ vanishes only at $p = 0$.

$$k = 2 \Leftrightarrow (c_\alpha(x))_{|\alpha|=2} \text{ is a symmetric postive definite matrix, for any } x \in \mathbb{R}^3.$$
$$\Leftrightarrow (c_\alpha(x))_{|\alpha|=2} \text{ yields a Riemannian metric } g \text{ on } \mathbb{R}^3.$$

The Hamiltonian flow of an elliptic operator is a co-geodesic flow of a Riemannian metric g. ($x(t)$ is a geodesic of g)

Scalar waves and travel times on a smooth bounded domain $M \subset \mathbb{R}^3$

Scalar wave equation: P is elliptic second order operator

$$\begin{cases} \left(\frac{\partial^2}{\partial x_0^2} - P(x, \mathrm{D}) + \text{l.o.t}\right) u(x_0, x) = 0, \quad (x_0, x) \in (0, T) \times M \\ u = h, \quad \text{on } (0, T) \times \partial M, \quad u(0, x) = \frac{\partial}{\partial x_0} u(0, x) = 0, \quad x \in M. \end{cases}$$

- **Principal symbol:** $G_{\Box}((x_0, x); (\omega, p)) = \omega^2 - G(x, p) = \omega^2 - g^{ij}(x) p_i p_j$
- **Bi-characteristics:** $(X(t), P(t)) = ((2\omega t, x(t)), (1, \pm p(t))) \in T^*\mathbb{R}^{1+3}$, ($x(t)$ is a geodesic of g).

Data:

- Let $\gamma(t) = (x(t), p(t))$ be an integral curve of G, $\|p\|_g = 1$, that enters M at x_1 and exits at x_2.
- **Travel time:** $d(x_1, x_2) := \inf_\gamma \{\inf\{t \in (0, \infty) : \text{ s.t. } x(0) = x_1,\ x(t) = x_2\}\}$ (Distance w.r.t. g).

Problem

Can we recover g if the travel time is given for all integral curves of G, passing through M?

Boundary rigidity (travel time) problem

Assumption (data):

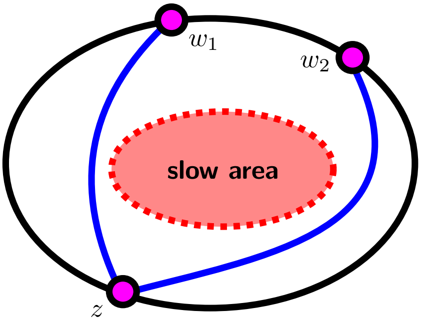
- $(M, g_i), i \in \{1, 2\}$ smooth Riemannian manifolds with boundary

$$d_1(z, w) = d_2(z, w); \quad z, w \in \partial M$$

Inverse Problem: Is there a Riemannian isometry $\Psi \colon (M, g_1) \to (M, g_2)$ that is an identity at ∂M?

(invariance of the problem) **If so** (M, g_i) **is boundary rigid**

In general, manifolds are not boundary rigid



Small variations of the metric in the slow area won't affect boundary distances $d(z, w_i)$!

"Better geometries"

- Simple manifolds:
  - ∂M is strictly convex
  - Each pair of points is connected by a unique minimizing geodesic

Conjecture: Simple Riemannian manifolds are boundary rigid (Michel 1981)

- 2D (Pestov-Uhlmann 2005)

- Convex foliation:
  - $\{f^{-1}\{s\} \subset M$ strictly convex surface : $\forall s \in [0, S)\}$
  - smooth $f : M \to [0, S)$, $Df \neq 0$

Foliated $3D$ Riemannian manifolds are boundary rigid (Stefanov-Uhlmann-Vasy 2017)

Small data (sources only at the boundary) makes boundary rigidity problem very difficult!

$$\Rightarrow$$

- more data (broken geodesics)
- more general geometry

Contents

Elastic wave equation on a bounded smooth domain $M \subset \mathbb{R}^3$

$T > 0, \quad \mathbf{C}(x) = C^{j\ell}_{ik}(x) = C^{i\ell}_{jk}(x) = C^{\ell j}_{ki}(x)$ anisotropic stiffness tensor on M.

Anisotropic elastic wave equation in Cartesian coordinates:

$$\begin{cases} \left(\delta_{ik}\frac{\partial^2}{\partial t^2} - \left(C^{j\ell}_{ik}(x)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^\ell}\right) + \text{l.o.t}\right) u^k(t,x) = 0, \quad (t,x) \in (0,T) \times M \\ u = h, \quad \text{on } (0,T) \times \partial M, \quad u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0, \quad x \in M. \end{cases}$$

displacement: u, boundary source: $h \in C^\infty((0,T) \times \partial M)$,

Principal symbol: $\omega^2 \delta_{ik} - \Gamma_{ik}(x,p), \quad i,k \in \{1,2,3\}$.

Christoffel matrix: $\Gamma_{ik}(x,p) := C^{j\ell}_{ik}(x) p_j p_\ell$ is symmetric in (i,k).

Travel time tomography for qP-waves.

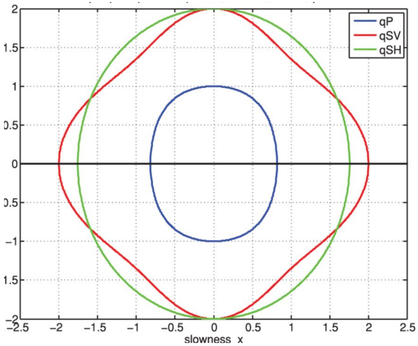
Γ_{ik} has 3 eigenvalues $G^m \in C(T^*\mathbb{R}^3)$.

Suppose $G^1(x,p) > G^m(x,p),\ m \in \{2,3\}$.

qP-Hamiltonian:

$$H(x,p) := G^1(x,p) \in C^\infty(T^*\mathbb{R}^3).$$

is given by a Finsler metric.



qP-waves are the solutions of ΨDO equation: $\left(\frac{\partial^2}{\partial t^2} - G^1(x,\mathrm{D})\right)u(t,x) = 0, \quad$ in $(0,T)\times M$.

Problem

Can we recover G^1 if the travel times for all qP-waves, passing through M, are known?

What is a Finsler manifold?

Let M be a smooth manifold of dimension $n \geq 2$

A continuous function $F\colon TM \to [0,\infty)$ is a Finsler metric if

1. $F\colon TM \setminus \{0\} \to [0,\infty)$ is smooth
2. for all $(x,v) \in TM$ and $a > 0$ holds $F(x,av) = aF(x,v)$
3. for all $(x,v) \in TM \setminus \{0\}$ the Hessian

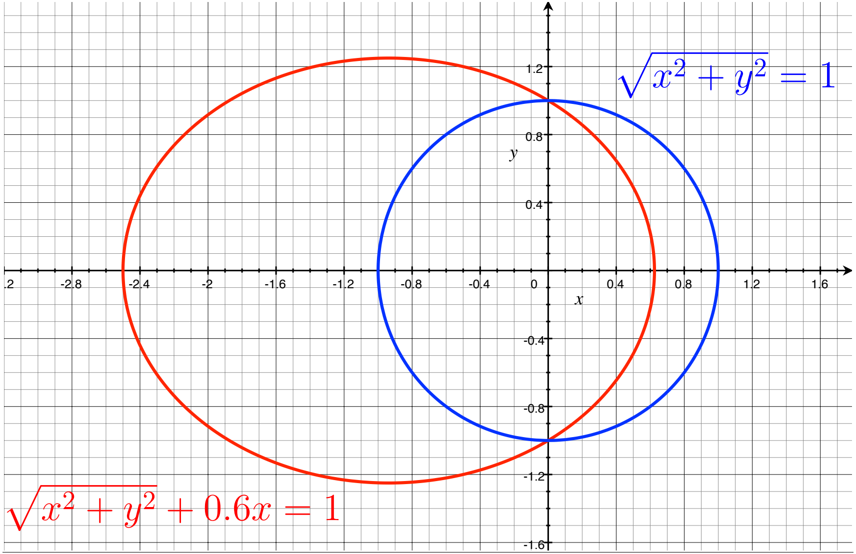
$$\left(\frac{1}{2}\frac{\partial}{\partial v^i}\frac{\partial}{\partial v^j}F^2(x,v)\right)_{i,j=1}^n =: \left(g_{ij}(x,v)\right)_{i,j=1}^n$$

   is symmetric and positive definite, g_{ij} is called the local Riemannian metric

$F(x,\cdot)$ is a non-symmetric $(F(x,v) \neq F(x,-v))$ norm on every tangent space T_xM, $x \in M$

Finsler function F is Riemannian if and only if $g(x,v) = g(x)$

Euclidean and Randers unit spheres



The Finsler distance is defined as Riemannian, by the length of curves

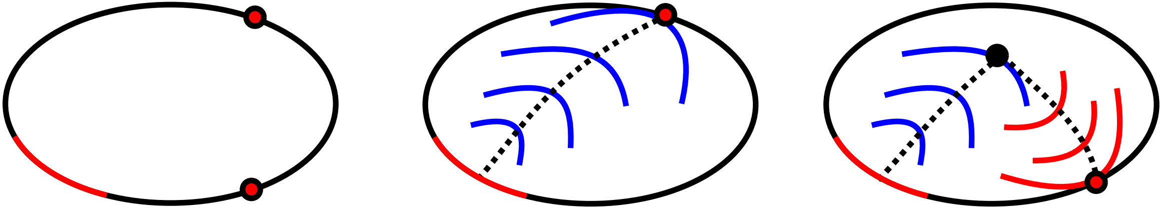
In general Finsler distance function is not symmetric

$$d(p,q) \neq d(q,p), \quad p,q \in M$$

However we assume that: $d(p,q) = d(q,p)$.

Contents

Reflection tomography for qP-waves



Problem

Can we recover the largest eigen value G^1 if we know

- *total travel times*
- *entering points and directions*
- *exiting points and directions*

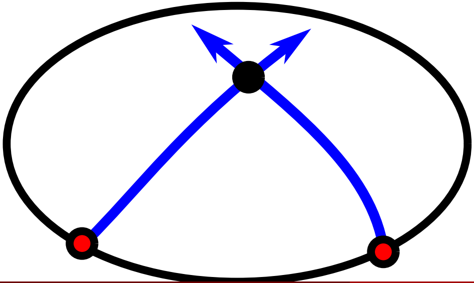
For any qP-wave that scatters once?

Broken ray tomography on Finsler manifolds

Definition (Broken scattering relation)

Let (M, F) be a compact symmetric Finsler manifold with boundary. For each $t > 0$ we define a relation R_t on $\partial_{\text{in}}\Omega M$ so that $(x_1, v_1)R_t(x_2, v_2)$ if there exist two numbers $t_1, t_2 > 0$ for which

$$t_1 + t_2 = t \quad \text{and} \quad \gamma_{x_1,v_1}(t_1) = \gamma_{x_2,v_2}(t_2).$$



Problem

Can we recover (M, F), modulo change of coordinates, from $\{R_t : t > 0\}$?

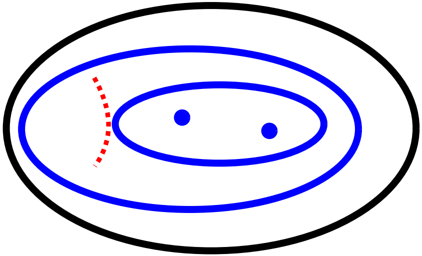
- *Yes, if F is Riemannian, Kurylev-Lassas-Uhlmann (2010)*

Convex foliation condition

Definition

A Finsler manifold with boundary has a strictly convex foliation if there is a smooth function $f\colon M \to \mathbb{R}$ so that

1. $f^{-1}\{0\} = \partial M$, $f^{-1}(0,S] = \mathrm{int}(M)$, $f^{-1}(S)$ has empty interior
2. for each $s \in [0,S)$ the set $f^{-1}(s)$ is a strictly convex smooth surface



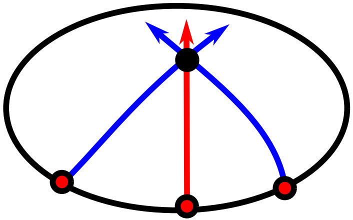
For radial wave speed $F = c(r)\delta_{ij}$ this is a generalization of the classical Herglotz condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0,$$

Theorem (de Hoop, Ilmavirta, Lassas, S (2020))

Let (M, F) be a compact symmetric Finsler manifolds of $\dim \geq 3$, with boundary. If $(\partial M, F|_{\partial TM})$ is known and (M, F) has a strictly convex foliation, then the broken scattering relations $\{R_t : t > 0\}$ determine (M, F) modulo change of coordinates.

Proof:



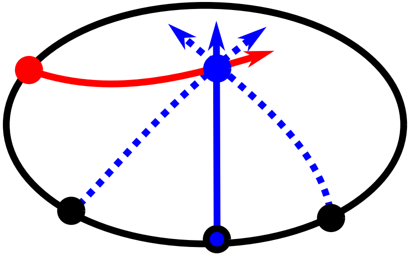
- Any $p \in M$ can be represented (non-uniquely) by its closest point $z_p \in \partial M$ and $d(p, z_0)$ distance to ∂M
- For $(z_0, t_0) \in \partial M \times [0, \tau_{\partial M}(z_0)]$ find the rays $\gamma_{z,V(z)}$

$$F(x_0, t_0) = \{(V(z), t(z)) : z \in \partial M, \text{ near } z_0\}$$

crossing the normal ray at $\gamma_{z_0,\nu}(t_0) = \gamma_{z,V(z)}(t(z))$.

- $V(z_1)R_{t(z_1)+t(z_2)}V(z_2)$ only tells that two rays intersect, not the intersection point!

From broken Scattering relation to boundary distance functions



- Choose $z_0, q \in \partial M$ and $t_0 \in [0, \tau_{\partial M}(z_0)]$. Denote $p := \gamma_{z_0,\nu}(t_0)$
- ∂M convex $\Rightarrow$ a distance minimizing curve $\gamma_{q,\eta} : [0, d(p,q)] \to M$ from q to p is a geodesic.

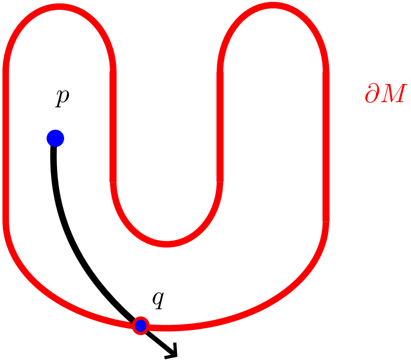
$$S := \{s > 0 : \text{ There exist } \eta \in S_q M, \text{ and } F(z_0, t_0) \text{ s.t. } V(z) R_{s+t(z)} \eta \text{ holds}\}$$

Then $d(p,q) = \inf S$

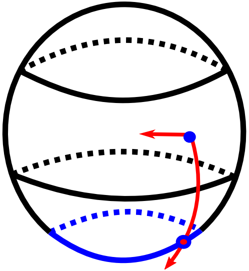
Finslerian boundary distance functions

Boundary distance function:

$$r_p \colon \partial M \to \mathbb{R}, \quad r_p(q) := d(p,q).$$



Boundary distance data: $(\partial M, \{r_p : p \in M^{int}\})$



$G(M,F)$ the set of directions corresponding to distance minimizing geodesic to ∂M.

Theorem (de Hoop, Ilmavirta, Lassas, S)

The boundary distance data determine

- *topology and coordinate structure of M.*
- *F in $G(M,F)$, but not outside.*

Using the foliation to peal M

We have the foliation given by function $f\colon M \to \mathbb{R}$ so that

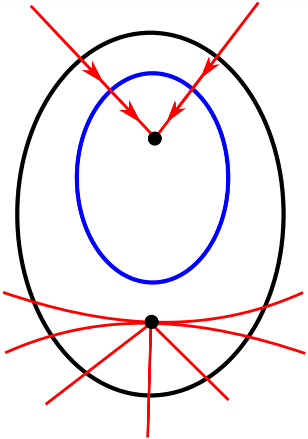
1. $f^{-1}\{0\} = \partial M$, $f^{-1}(0,S] = \text{int}(M)$, $f^{-1}(S)$ has empty interior
2. for each $s \in [0,S)$ the set $f^{-1}(s)$ is a strictly convex smooth surface.

Pealing procedure:

- Since ∂M is convex there exists $\epsilon > 0$ s.t.

  $$d(p,\partial M) < \epsilon \;\Rightarrow\; \text{hemisphere at } p \subset G(M,F).$$

- We know F in $G(M,F)$
- Symmetry $\Rightarrow$ we know F on T_pM, if $d(p,\partial M) < \epsilon$.
- Recover F on $f^{-1}[0,s]$, $s > 0$ depending on ϵ.
- Push data on the convex surface $f^{-1}(s)$.
- Repeat the arguments above, until M has been depleted.



Talk was based on the following manuscripts

1. **A foliated and reversible Finsler manifold is determined by its broken scattering relation**, in collaboration with, Maarten V. de Hoop, Joonas Ilmavirta and Matti Lassas, arXiv:2003.12657

2. **Inverse problem for compact Finsler manifolds with the boundary distance map**, in collaboration with, Maarten V. de Hoop, Joonas Ilmavirta and Matti Lassas, arXiv:1901.03902

Thank you for your attention!